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THE EFFECT OF STRAIN-HARDENING ON THE DEFORMATION OF AN ANNULAR SLAB

by

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The Effect of Strain-hardening on the Deformation of an Annular Slab[1]

by

P. G. Hodge, Jr.[2]

A procedure is outlined for obtaining the stresses and strains in a circular slab with a cutout, subject to uniform biaxial tension. An arbitrary stress-strain curve in tension is approximated by any number of straight line segments. For biaxial states of stress the material is assumed to satisfy a flow law based on the maximum shear stress, and to be incompressible throughout. The general equations are given and then simplified by assuming that boundary motions may be neglected if the strains are small, and that elastic strain components may be neglected if the strains are large. For the case of linear strain-hardening a complete solution is given in closed form. If the rate of strain-hardening is small, these results may be further simplified.

[1]The results presented in this paper were obtained in the course of research conducted as a Consultant under Contract N7onr-35810 between the Office of Naval Research and Brown University.

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INTRODUCTION

Consider an annular slab of an incompressible material of uniform thickness H_0, subjected to a uniform biaxial tension of magnitude $EH_0\lambda$ on the outer edge (Fig. 1). It is assumed that the slab is in a state of generalized plane stress, even though the thickness may vary as the deformation progresses. Due to symmetry, the mechanical state of the slab may be specified in terms of the following four functions of position and time:

$$\left.\begin{array}{lll} \sigma = \sigma_r(r,t)/E & \text{reduced radial stress,} \\ s = \sigma_\theta(r,t)/E & \text{reduced circumferential stress,} \\ u = u_r(r,t) & \text{radial displacement,} \\ h = h\,(r,t) & \text{thickness,} \end{array}\right\} \quad (1)$$

where E is Young's modulus.

Regardless of the relations between stress and strain, the slab must satisfy the equation of equilibrium

$$\frac{\partial}{\partial r}\,(rh\sigma) - hs = 0, \qquad (2)$$

and the condition of incompressibility

$$r\,\frac{\partial h}{\partial t} + \frac{\partial}{\partial r}\,(rhv) = 0. \qquad (3)$$

Here v is the particle velocity:

$$v = \left(1 + \frac{\partial u}{\partial r}\right)\frac{\partial u}{\partial t}.$$

So long as the load is increasing, the material will be assumed to be elastic provided that the maximum shearing stress is less than half the initial yield stress in simple tension. It may be verified a posteriori that the principal stresses satisfy

$$\sigma_\theta \geq \sigma_r \geq \sigma_z \equiv 0, \qquad (4)$$

so that the material is elastic provided

$$f(\sigma, s) = s - \mu < 0, \tag{5}$$

where μ is the reduced initial yield stress in simple tension. If Inequality 5 is valid, then two additional equations are furnished by Hooke's law:

$$\left.\begin{aligned} \varepsilon_r &= \partial u/\partial r = (2\sigma_r - \sigma_\theta)/2E = \sigma - s/2, \\ \varepsilon_\theta &= u/r = (2\sigma_\theta - \sigma_r)/2E = s - \sigma/2. \end{aligned}\right\} \tag{6}$$

The plastic domain is characterized by the relations

$$s - \mu \geq 0, \quad \dot{s} \geq 0.$$

The general form of flow law obtained by Prager and Hodge [1][3] may be written in terms of the reduced principal stress components

$$\dot{\varepsilon}_1'' = F \frac{\partial f}{\partial \sigma_1'} \dot{f}$$

and two similar expressions for $\dot{\varepsilon}_2''$ and $\dot{\varepsilon}_3''$. Here $\dot{\varepsilon}_1''$, $\dot{\varepsilon}_2''$, $\dot{\varepsilon}_3''$ are the plastic strain rate components, σ_1', σ_2', σ_3' are the reduced stress components, f is the yield function, and F is a function of stress to be determined from the stress-strain curve. If f is given as in Eq. 5, the stress-strain law becomes

$$\dot{\varepsilon}_r'' = 0, \; \dot{\varepsilon}_\theta'' = F\,\dot{s}. \tag{7}$$

Let the given stress-strain curve be approximated by a series of straight line segments, (Fig. 2), and let α_k be the inclination of the k^{th} segment in the strain-reduced stress plane. In simple tension, the relation between plastic strain rate and stress rate is then given by

$$E\dot{\varepsilon}_x'' = \cot \alpha_k \, \dot{\sigma}_k \; , \; \mu_k \leq \sigma_x/E \leq \mu_{k+1} \; ,$$

[3] Numbers in brackets refer to references collected at the end of the paper.

so that Eq. 7 becomes

$$\dot{\varepsilon}_r'' = 0, \quad \dot{\varepsilon}_\theta'' = \cot \alpha_k \, \dot{s}, \quad \mu_k \leq s \leq \mu_{k+1}.$$

Finally, since the elastic components of strain rate must satisfy the differentiated form of Hooke's law, the plastic stress-strain relations are

$$\left.\begin{aligned} \dot{\varepsilon}_r &= \partial \dot{u}/\partial r = \dot{\sigma} - \dot{s}/2, \\ \dot{\varepsilon}_\theta &= \dot{u}/r = (1+\cot \alpha_k)\dot{s} - \dot{\sigma}/2, \end{aligned}\right\} \mu_k \leq s \leq \mu_{k+1}. \tag{8}$$

Equations 2, 3, and 6 or 8 furnish four equations for the four unknowns σ, s, h, and u. At the boundaries, the stresses must be in equilibrium with the applied tractions, so that

$$\sigma(a) = 0, \tag{9}$$

$$h(d)\sigma(d) = H_o\lambda. \tag{9b}$$

In addition, all stresses and strains must be continuous throughout the slab, so that if $r = \rho_k$ is the radius which divides one analytic form of solution from another,

$$\begin{aligned} \sigma(\rho_k^-) &= \sigma(\rho_k^+), \quad s(\rho_k^-) = s(\rho_k^+), \\ u(\rho_k^-) &= u(\rho_k^+), \quad \frac{\partial u}{\partial r}(\rho_k^-) = \frac{\partial u}{\partial r}(\rho_k^+). \end{aligned} \tag{10}$$

ELASTIC SOLUTION

Since h and u are both unknown, Eqs. 2 and 3 are non-linear, and at present there does not exist a general solution to the boundary value problem defined in the previous section. However, as long as the strains are small compared to one, it appears reasonable to neglect motion of the boundaries and changes of the thickness h. For a perfectly plastic material[2] it is known that the errors introduced by these approximations are of order μ~0.001.

Regarding h as constant, Eq. 3 can no longer be satisfied exactly, and Eqs. 2 and 6 become linear. The solution for an elastic material is readily found to be

$$\left.\begin{array}{l} u = Br + C/r, \\ \left.\begin{array}{l}\sigma \\ s\end{array}\right\} = 2\,(B \mp C/3r^2) \end{array}\right\}, \qquad (11)$$

where B and C are to be determined from the boundary conditions. For λ sufficiently small, the slab will be elastic throughout and the boundary conditions 9 may both be applied to determine the constants. Since boundary motions are neglected, the initial values of a and d may be used. The resulting solution is

$$\left.\begin{array}{r} u/r \\ \partial u/\partial r \end{array}\right\} = \frac{\lambda}{2}\,\frac{1 \pm 3a_o^2/r^2}{1 - a_o^2/d_o^2},$$

$$\left.\begin{array}{l}\sigma \\ s\end{array}\right\} = \lambda\,\frac{1 \mp a_o^2/r^2}{1 - a_o^2/d_o^2}. \qquad (12)$$

The above solution will be valid so long as Inequality 5 is satisfied throughout the slab. Since s is a maximum at the inner boundary, the maximum load for which this is true is found by setting $s(a) = \mu$ and solving for λ:

$$\lambda_1 = \tfrac{1}{2}\,\mu\,(1 - a_o^2/d_o^2). \qquad (13)$$

For values of λ somewhat larger than λ_1, a plastic ring will form against the inner boundary. If ρ denotes the radius between the elastic and plastic regions, then the slab is still elastic for $\rho \leq r \leq d_o$. In this region, Eqs. 11 are still valid, but the boundary condition 9a is no longer applicable. In its stead, the condition

$$s(\rho) = 2(B+C/3\rho^2) = \mu$$

must be satisfied. Solving this for C and substituting into Eqs. 11 one obtains the elastic solution in the form

$$\left.\begin{aligned} \left.\begin{matrix} u/r \\ \partial u/\partial r \end{matrix}\right\} &= B(1 \mp 3 \frac{\rho^2}{r^2}) \pm \frac{3}{2}\mu \frac{\rho^2}{r^2}, \\ \left.\begin{matrix} \sigma \\ s \end{matrix}\right\} &= 2B(1 \pm \frac{\rho^2}{r^2}) \mp \mu \frac{\rho^2}{r^2}. \end{aligned}\right\} \quad (14)$$

The boundary condition at $r = d$ will be used to determine a relationship between λ and ρ, once the constant B has been determined from the plastic solution.

ELASTIC-PLASTIC SOLUTION-FIRST STAGE

For values of the applied load slightly greater than λ_1, the plastic portion of the slab will correspond entirely to the first plastic segment AB of the stress-strain curve (Fig. 1). So long as part of the slab is still elastic, the plastic stress-strain laws (Eqs. 8) may be integrated to yield

$$\begin{aligned} \varepsilon_r &= \partial u/\partial r = \sigma - s/2, \\ \varepsilon_\theta &= u/r = (1+\cot\alpha)s - \sigma/2 - \mu\cot\alpha. \end{aligned} \quad (15)$$

The constants of integration were determined from the fact that when a given particle first enters the plastic domain, the strains must be given by Eqs. 6 and s must equal μ.

If Eqs. 15 are solved for s and σ in terms of u, and the results substituted into the equilibrium equation, the resulting linear equation for u is readily solved to yield

$$u/r = Kr^{n-1} + Lr^{-n-1} + \tfrac{1}{2}\mu, \quad (16a)$$

$$\partial u/\partial r = nKr^{n-1} - nLr^{-n-1} + \tfrac{1}{2}\mu, \quad (16b)$$

$$\sigma = \frac{2nK}{2-n} r^{n-1} + \frac{2nL}{2+n} r^{-n-1} + \mu, \quad (16c)$$

$$s = \frac{2n^2K}{2-n} r^{n-1} + \frac{2n^2L}{2+n} r^{-n-1} + \mu. \tag{16d}$$

Here n is defined by

$$n = (1+\cot \alpha)^{-1/2}, \tag{17}$$

and functions of α have been replaced by functions of n throughout.

The four functions given in the plastic region by Eqs. 16 and in the elastic region by Eqs. 14 must be continuous at the elastic-plastic boundary $r = \rho$. Together with Eq. 9a, this provides five equations to determine the three constants B, K, and L. However, in view of the continuity imposed in integrating the plastic stress-strain law, two of these equations are redundant. After some computation it is found that

$$\left.\begin{aligned} B &= \tfrac{1}{2}(1- 1/\Delta)\mu, \\ K &= \frac{2+n}{2n} \frac{\mu}{\Delta} \rho^{1+n}, \\ L &= \frac{2-n}{2n} \frac{\mu}{\Delta} \rho^{1-n}, \end{aligned}\right\} \tag{18}$$

where

$$\Delta = (\rho/a_o)^{1+n} + (\rho/a_o)^{1-n}. \tag{19}$$

The substitution of Eqs. 18 into Eqs. 16 and 14 then yields the following complete solution:

Plastic ($a_o \leq r \leq \rho$):

$$\frac{u}{r} = \tfrac{1}{2}\mu\left[1 + \frac{2+n}{n\Delta}\left(\frac{\rho}{r}\right)^{1+n} - \frac{2-n}{n\Delta}\left(\frac{\rho}{r}\right)^{1-n}\right], \tag{20a}$$

$$\frac{\partial u}{\partial r} = \tfrac{1}{2}\mu\left[1 - \frac{2+n}{\Delta}\left(\frac{\rho}{r}\right)^{1+n} - \frac{2-n}{\Delta}\left(\frac{\rho}{r}\right)^{1-n}\right], \tag{20b}$$

$$\sigma = \mu\left[1 - \frac{1}{\Delta}\left(\frac{\rho}{r}\right)^{1+n} - \frac{1}{\Delta}\left(\frac{\rho}{r}\right)^{1-n}\right], \tag{20c}$$

$$s = \mu\left[1 + \frac{n}{\Delta}\left(\frac{\rho}{r}\right)^{1+n} - \frac{n}{\Delta}\left(\frac{\rho}{r}\right)^{1-n}\right]; \tag{20d}$$

Elastic ($\rho \leq r \leq d_o$):

$$\left.\begin{matrix} u/r \\ \partial u/\partial r \end{matrix}\right\} = \tfrac{1}{2}\mu\left[1 - \frac{1 \mp 3\rho^2/r^2}{\Delta}\right], \tag{20e,f}$$

$$\left.\begin{matrix} \sigma \\ s \end{matrix}\right\} = \mu\left[1 - \frac{1 \pm \rho^2/r^2}{\Delta}\right]. \tag{20g,h}$$

Equations 20 give the complete solution in terms of the position of the elastic-plastic boundary. The load necessary to extend the plastic region to a given boundary ρ is obtained from the boundary condition, Eq. 9b:

$$\lambda = \sigma\ (d_o) = \mu\left[1 - \frac{1+\rho^2/d_o^2}{\Delta}\right]. \tag{20i}$$

Since the parameter n occurs as an exponent in Eqs. 20, the preceding solution is not well adapted to computations. For most real materials n is small compared to one, so that it appears reasonable to approximate the solution and obtain a more useful formulation. This may be done by expanding the solution as power series in n. Since all the quantities are even functions of n, the following expressions are correct up to terms n^4:

Plastic ($a_o \leq r \leq \rho$):

$$\frac{u}{r} = \tfrac{1}{2}\mu\Big\{1 + \frac{a_o}{r} + 2\,\frac{a_o}{r}\log\frac{\rho}{r} + \frac{a_o}{r}\,\frac{n^2}{6}\Big[(3+2\log\frac{\rho}{r})(\log\frac{\rho}{r})^2 - 3(1+2\log\frac{\rho}{r})(\log\frac{\rho}{a_o})^2\Big]\Big\}, \tag{21a}$$

$$\frac{\partial u}{\partial r} = \tfrac{1}{2}\mu\Big\{1 - 2\,\frac{a_o}{r} - n^2\log\frac{a_o}{r}\Big[\log\frac{\rho}{r} + (\log\frac{\rho}{r})^2 - (\log\frac{\rho}{a_o})^2\Big]\Big\}; \tag{21b}$$

$$\sigma = \mu\Big\{1 - \frac{a_o}{r} - \tfrac{1}{2}n^2\,\frac{a_o}{r}\Big[(\log\frac{\rho}{r})^2 - (\log\frac{\rho}{a_o})^2\Big]\Big\}, \tag{21c}$$

$$s = \mu\Big\{1 + n^2\,\frac{a_o}{r}\log\frac{\rho}{r}\Big\}; \tag{21d}$$

Elastic ($\rho \leq r \leq d_o$):

$$\left.\begin{matrix} u/r \\ \partial u/\partial r \end{matrix}\right\} = \tfrac{1}{2}\mu\left\{1 - \frac{a_o}{2\rho}\left(1 \mp 3\,\frac{\rho^2}{r^2}\right)\left[1 - \frac{n^2}{2}\left(\log\frac{\rho}{a_o}\right)^2\right]\right\}, \qquad (21e,f)$$

$$\left.\begin{matrix} \sigma \\ s \end{matrix}\right\} = \mu\left\{1 - \frac{a_o}{2\rho}\left(1 \pm \frac{\rho^2}{r^2}\right)\left[1 - \frac{n^2}{2}\left(\log\frac{\rho}{a_o}\right)^2\right]\right\}; \qquad (21g,h)$$

Load:

$$\lambda = \mu\left\{1 - \frac{a_o}{2\rho}\left(1 + \frac{\rho^2}{d_o^2}\right)\left[1 - \frac{n^2}{2}\left(\log\frac{\rho}{a_o}\right)^2\right]\right\}. \qquad (21i)$$

ELASTIC-PLASTIC SOLUTION-SECOND STAGE

The solution given by Eqs. 20 (or approximately by Eqs. 21) is valid so long as the following two inequalities are both satisfied:

$$\rho \leq d_o, \quad s(a) \leq \nu \;. \qquad (22)$$

As the load is increased so that one or the other of these inequalities is violated, a variety of behavior may be exhibited depending upon the dimensions of the slab and the parameters in the stress-strain curve.

Consider first the case where these quantities are such that the second Inequality 22 is violated while the first remains valid, and lot ρ_1 be the value of ρ for which this occurs. To the approximations used in Eq. 21d it follows that ρ_1 is given by

$$\mu\;(1+n^2 \log \rho_1/a_o) = \nu. \qquad (23)$$

Since by hypothesis ρ_1 is less than d_o the parameters of the problem must satisfy

$$\mu\;(1+n^2 \log d_o/a_c) > \nu. \qquad (24)$$

The approximate load λ_2 for which this first occurs is obtained by solving Eq. 23 for ρ_1 and substituting the result into Eq. 21i.

For values of λ somewhat greater than λ_2 the solution in the slab will consist of three regions. If ζ represents the radius where $s = v$, then for $a_o \leq r \leq \zeta$ the material is plastic corresponding to the segment BC of the stress-strain curve (Fig. 2); for $\zeta \leq r \leq \rho$ the material is plastic corresponding to AB; and for $\rho \leq r \leq d_o$, the material is elastic, corresponding to OA. For the latter two regions, the solution is given by Eqs. 17 and 14, respectively, where the boundary constants must be determined anew.

For the innermost region, the stress-strain law Eq. 8 may again be integrated to yield

$$\left.\begin{aligned} \varepsilon_r &= \partial u/\partial r = \sigma - s/2, \\ \varepsilon_\theta &= u/r = (1 + \cot\beta)s - \sigma/2 - v(\cot\beta - \cot\alpha) - \mu\cot\alpha \end{aligned}\right\} \qquad (25)$$

As in the previous section, these relations may be solved for σ and s and the results substituted into the equilibrium equation. The resulting equation is readily solved for u:

$$\frac{u}{r} = Pr^{m-1} + Qr^{-m-1} + \frac{1}{2}(\mu - v)\frac{1-n^2}{1-m^2}\frac{m^2}{n^2} + \frac{1}{2}v, \qquad (26)$$

where

$$m = (1 + \cot\beta)^{-1/2}. \qquad (27)$$

The constants P, Q, K, L, and B may now be determined from the continuity conditions (Eqs. 10) at ζ and ρ. The boundary condition 9a then determines the relation between ρ and ζ, while Eq. 9b furnishes the value of the load. The computations and results become quite complicated and are not reproduced here.

Consider now the other possibility characterized by the reverse of Inequality 24. In this case the slab will become fully plastic for a load λ_2' obtained by setting ρ equal to d_o in Eq. 20i or 21i. Thus from Eq. 21i,

$$\lambda_2' = \mu \left[1 - \frac{a_o}{d_o} + \frac{n^2}{2} \frac{a_o}{d_o} (\log \frac{d_o}{a_o})^2 \right] . \tag{28}$$

For values of λ slightly greater than λ_2', Eqs. 16 will be valid throughout the slab. The constants K and L may be determined directly in terms of the load λ by substituting Eq. 16c into Eqs. 9. While the results may be obtained accurately, it is somewhat simpler to introduce the approximate power series in n directly. After some computation, the complete solution is found to be

$$\left.\begin{aligned}
\frac{u}{r} &= \frac{1}{n^2} \frac{d_o}{r} \frac{(a_o/d_o)\mu + \lambda - \mu}{\log (d_o/a_o)}, \\
\frac{\partial u}{\partial r} &= \frac{d_o/r}{2 \log (d_o/a_o)} \left[-\mu \frac{a_o}{d_o} (1+2 \log \frac{d_o}{r}) + (\lambda-\mu)(2 \log \frac{r}{a_o} - 1) \right] + \tfrac{1}{2} \mu , \\
\sigma &= \frac{d_o/r}{\log (d_o/a_o)} \left[-\mu \frac{a_o}{d_o} \log \frac{d_o}{r} + (\lambda-\mu) \log \frac{r}{a_o} \right] + \mu , \\
s &= \frac{d_o}{r} \frac{(a_o/d_o)\mu + \lambda - \mu}{\log (d_o/a_o)} + \mu ,
\end{aligned}\right\} \tag{29}$$

where only the leading terms in n^2 have been retained.

Equations 29 will be valid until $s(a) = \nu$, assuming that the strains may still be considered small. On the other hand, assuming for simplicity that the entire stress-strain curve is approximated by the three segments in Fig. 2, the solution obtained from Eqs. 27, 17, and 14 is valid until $\rho = d_o$. For

slightly greater loads in either case, the slab will be plastic throughout, but will be governed by the portion BC of the stress-strain curve for $a_o \leq r \leq \zeta$; and by AB for $\zeta \leq r \leq d_o$. Therefore, the solution will consist of Eqs. 27 for $a_o \leq r \leq \zeta$ and Eqs. 17 for $\zeta \leq r \leq d_o$, with the four constants P, Q, K, and L to be determined from Eqs. 9 and 10.

If a more accurate approximation to the stress-strain curve is desired, the solution may be obtained in similar fashion, although the resulting computations would become exceedingly laborious.

FULLY PLASTIC SOLUTION-LARGE STRAINS

The solutions obtained in the preceding sections are valid only as h may be considered a constant and the boundary displacements neglected. For an incompressible material these conditions will both be satisfied provided that the strains are small compared to one throughout the slab. This will generally be the case so long as any part of the slab is elastic, at least provided a_o/d_o is not too small. However, once the slab is fully plastic, relatively large strains may occur and the previous solutions would no longer be applicable.

Since h can no longer be considered constant, the equations to be solved are non-linear, and it is necessary to make some other simplifying assumption to obtain a solution. The elastic part of the strain remains proportional to the stress throughout loading and hence can never become large. Therefore, if the total strains are large, it appears reasonable to neglect entirely the elastic component and consider the strains as purely plastic.

This is equivalent to assuming that Young's modulus is equal to infinity, so that the elastic portion of the stress-strain curve in Fig. 2 is replaced by a vertical line. For simplicity of exposition it will be assumed that the plastic portion of the stress-strain curve consists of a single linear segment (Fig. 3) although the method can be extended to more general curves as in the previous sections.

In the case of small deformations, it was not necessary to distinguish between the Lagrangian (original) and Eulerian (current) coordinates of a particle. However, for the large strains considered in the present section such a distinction is obligatory. The stress-strain law is a function of the stress and strain of the particular particle in question, and hence must be expressed in terms of the original coordinate r_o of a particle now at r . Thus, since elastic strain rates are neglected, Eqs. 8 are replaced by

$$\dot{\varepsilon}_r = \partial w/\partial r = o, \tag{30a}$$

$$\dot{\varepsilon}_\theta = \frac{v}{r} = \cot \alpha \, \dot{s} = [\,(1-n^2)/n^2\,]\, \dot{s} \ . \tag{30b}$$

The "dot" in Eqs. 30 customarily refers to time. However, since inertia effects are not considered, time may be replaced by any convenient monotonic function of time. In the present case, the current radius of the inner boundary is such a function.[4]

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[4] The validity of this step may be seen by noting that the rate at which load is applied will not affect the solution. Thus, in particular, the load may be applied so that the inner boundary moves outward at a uniform rate. Finally, the unit of time may be chosen so that this rate is unity.

Therefore Eq. 30a states that

$$\frac{\partial}{\partial r} \; v\,(r,a) = 0.$$

While this equation may easily be integrated to obtain the velocity, it is more convenient to first transform it to the Lagrangian coordinate r_o . Since the "time" a occurs only as a parameter,

$$\frac{\partial v}{\partial r} = \frac{\partial v/\partial r_o}{\partial r/\partial r_o} = 0,$$

hence

$$\frac{\partial v}{\partial r_o} = \frac{\partial}{\partial r_o}\left(\frac{\partial u}{\partial a}\right) = 0,$$

where $u(r_o, a)$ is the displacement of the particle. The integral of this equation shows that

$$u\,(r_o,a) = F(r_o) + G(a).$$

Since elastic strains are neglected, the displacement of all particles is zero at the commencement of plastic flow. Thus

$$u\,(r_o,a_o) = F(r_o) + G(a_o) \equiv 0.$$

$F(r_o)$ can now be evaluated, the resulting displacement being

$$u\,(r_o,a) = G(a) - G(a_o).$$

In view of the choice of a time scale, the velocity of the particle initially at the inner boundary (and hence always at the current inner boundary) is unity, so that

$$\left.\frac{\partial u}{\partial a}\right|_{r_o = a_o} = \left. G'(a)\right|_{r_o = a_o} \equiv 1.$$

Therefore $G(a) = a$, and hence

$$u\,(r_o,a) = a - a_o \tag{31a}$$

independently of r_o . Further the current position of the particle initially at r_o is

$$r(r_o,a) = r_o + u(r_o,a) = r_o + a-a_o. \tag{31b}$$

Finally, the velocity at all points is

$$v = 1. \tag{31c}$$

The substitution of Eqs. 31 into Eq. 30b yields

$$\frac{\partial s}{\partial a} = \frac{n^2}{1-n^2} \frac{1}{r_o+a-a_o},$$

hence

$$s(r_o,a) = \frac{n^2}{1-n^2} \log (r_o+a-a_o) + f(r_o). \tag{32}$$

The function $f(r_o)$ is to be evaluated from the continuity of s at $a = a_o$. However, since there has been a change of assumption between the small strain and large strain solutions, this continuity cannot be exact. The best one can do is to evaluate $s(r_o,a)$ so that it is continuous with the solution given by Eq. 20d for $\rho = d_o$. This leads, of course, to a rather awkward expression for $f(r_o)$. However, this expression may be simplified by neglecting terms of order n^4. This leads finally to

$$s = \mu + n^2 \left[\log \frac{r_o+a-a_o}{r_o} + \mu \frac{a_o}{r_o} \log \frac{d_o}{r_o} \right],$$

a result which could also have been obtained directly from Eq. 21d. Finally, since μ is very small compared to one, the second term in the coefficient of n^2 may be neglected in comparision with the first, once the deformation has progressed. Thus, to within the approximations considered,

$$s = \mu + n^2 \log \frac{r}{r_o}, \tag{33a}$$

where r and r_o are related by Eq. 31b. Equation 31d predicts that $s = \mu$ throughout the slab at the onset of full plasticity. Thus physically the assumptions are equivalent to neglecting

the strain-hardening which occurs during the elastic-plastic stage of loading. In an example considered later, this approximation appears to be well justified.

In Eq. 31c, the functional form of the velocity v is the same referred to either the Lagrangian or Eulerian coordinates. Thus the Eulerian form of Eq. 3 (incompressibility equation) may be retained:

$$\frac{\partial}{\partial a}(rh) + \frac{\partial}{\partial r}(rh) = 0,$$

where h is a function of r and a. The solution of this equation satisfying the condition $h = H_o$ when $a = a_o$ is

$$h = \left(1 - \frac{a-a_o}{r}\right) H_o = \frac{r_o}{r} H_o. \qquad (33b)$$

This result could also have been predicted directly as follows. For velocity constant, the thickness Δr of a narrow ring will remain constant, hence by incompressibility the volume $2\pi r h (\Delta r)$ must also remain constant at all times a.

The radial stress is then determined by integrating the equilibrium equation (Eq.2) with respect to the Eulerian coordinate r:

$$\sigma = \frac{1}{rh}\int_a^r hs\, dr$$

$$= \frac{1}{r_o}\int_a^r \left[\mu \frac{r_o}{r} + n^2 \frac{r_o}{r}\log\frac{r_o}{r}\right] dr$$

$$= \frac{\mu}{r_o}\left[r-a-(a-a_o)\log\frac{r}{a}\right] - \frac{n^2}{r_o}\int_a^r \frac{r_o}{r}\log\frac{r_o}{r}\, dr. \qquad (33c)$$

In the above integrations, r_o is defined as a function of r by Eq. 31b. Finally, the load necessary to enlarge the inner radius from a_o to any value a is obtained from Eq. 9b:

$$\lambda\ (a) = h\ (d,a)\ \sigma\ (d,a)/H_o$$

$$= \mu\ [1 - \frac{a}{d} - (\frac{a-a_o}{d})\ \log \frac{d}{a}]$$

$$- \frac{n^2}{d}\int_a^d \frac{r_o}{r} \log \frac{r_o}{r}\ dr,$$

where the current outer radius d is given by

$$d = d_o + a - a_o.$$

For a perfectly plastic material [2], it was found that the load λ <u>decreased</u> as the fully plastic deformation proceeded, so that the maximum load which the slab could support was achieved just as the elastic ring disappeared. For a material with infinite strain-hardening, on the other hand, the load could increase indefinitely above the fully plastic value. The question naturally arises as to the rate of strain-hardening which separates these two types of behavior. In other words, what is the minimum strain-hardening rate for which the initial fully-plastic configuration is stable?

Since λ is given as a function of a by Eq. 33d, this question is easily answered by differentiating λ with respect to a, setting $a = a_o$, and observing the sign of the derivative. If it is positive, then an increase in radius must be accompanied by an increase in load. After some computation, it is seen that the derivative at a_o is positive if and only if

$$n^2 > \mu\ \frac{1 - a_o/d_o + \log\ (d_o/a_o)}{\log\ d_o/a_o}. \qquad (34)$$

Equation 34 states the condition that the fully plastic solution be initally stable. As the deformation progresses, however, it follows from Eq. 33b that the thickness at the outer edge grows less, so that the same load produces a higher true stress. Therefore, for any finite rate of strain-hardening, the actual load which the slab can support will decrease after the deformation has proceeded sufficiently far. The value of a for which this is the case can be found by equating $d\lambda/da$ to zero and solving for a. Although this cannot be done in closed form, the maximum value of λ can be found numerically for any given values of the parameters.

EXAMPLE-PERFECTLY PLASTIC MATERIAL

A perfectly plastic material is characterized by a zero rate of strain-hardening, i.e., by $\alpha = 0$. Therefore, the complete solution for a perfectly plastic material would consist of three stages: fully elastic, elastic-plastic, and fully plastic. The elastic solution is still given by Eqs. 12, while in view of Eq. 17 the other two solutions are obtained by setting n=o in Eqs. 21, 31, and 33 respectively:

Elastic-plastic solution for $a_o \leq r \leq \rho$:

$$u/r = \tfrac{1}{2}\mu\,[1 + a_o/r + 2\,(a_o/r)\log(\rho/r)],$$

$$\partial u/\partial r = \tfrac{1}{2}\mu\,(1-2\,a_o/r),$$

$$\sigma = \mu\,(1-a_o/r),\quad s = \mu;$$

Elastic-plastic solution for $\rho \leq r \leq d_o$:

$$\left.\begin{matrix} u/r \\ \partial u/\partial r \end{matrix}\right\} = \tfrac{1}{2}\mu\,[1 - \frac{a_o}{2\rho} \pm 3\,\frac{a_o\rho}{2r^2}],$$

$$\left.\begin{matrix} \sigma \\ s \end{matrix}\right\} = \mu\,[1 - \frac{a_o}{2\rho} \mp \frac{a_o\rho}{2r^2}\,]\,; \qquad (35)$$

Elastic-plastic load:

$$\lambda = \mu \left[1 - \frac{a_o}{2\rho} - \frac{a_o\rho}{2d_o^2} \right];$$

Fully plastic solution:

$$\left.\begin{aligned} u/r &= (a-a_o)/r, \quad \partial u/\partial r = 0, \quad h = [\, 1-(a-a_o)/r \,]\, H_o, \\ \sigma &= \frac{r - a - (a-a_o)\log(r/a)}{r-a+a_o}, \quad s = \mu, \\ \lambda &= \mu \left[1 - a_o/d_o - (a/d_o - a_o/d_o)\log\left\{ (d_o + a - a_o)/a \right\} \right] \div \\ &\qquad (1 + a/d_o - a_o/d_o). \end{aligned}\right\} \quad (35)$$

In a previous paper [2] , the author obtained the complete perfectly plastic solution directly by a perturbation method. It may readily be verified that the first term of each quantity in the perturbation procedure agrees precisely with Eqs. 35.[5] In the discussion following the presentation of [2] at the 1952 Annual Meeting of A.S.M.E. (New York, N.Y., Dec. 2, 1952), the question was raised as to whether or not the perfectly plastic solution truly represented a limiting case of a strain-hardening solution; the present example shows that for the problem considered it does indeed.

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[5] The elastic-plastic solution is given by Eqs. 32 of [2] , and the fully plastic by Eqs. 39 through 46. Note that the dimensionless stresses there were defined by dividing by twice the shear modulus, 2G, rather than Young's modulus E, and that μ was defined by K/2G. Since E = 3G for an incompressible material, the results may be easily converted.

EXAMPLE-STRAIN-HARDENING MATERIAL[6]

Some results have been worked out for the following numerical data:

$\mu = 0.005$, $n^2 = 0.05$, $a_o/d_o = 1/3$.

The plastic part of the stress-strain curve has been approximated by a single linear segment. Figure 4 shows the load, circumferential strain $u(a_o)a_o$ of the inner boundary, and reduced circumferential stress $s(a_o)$ at the inner boundary, all as functions of the radius of the elastic-plastic boundary for the elastic-plastic solution given by Eqs. 21. The maximum strain, when the slab first becomes fully plastic, is seen to be greater than one per cent. Since this strain can be shown to increase rapidly with further increase in load, the fully plastic solution for small strains given by Eqs. 29 does not appear to be applicable for the present example. Therefore, in Fig. 5 the load, maximum stress, and maximum strain are plotted as functions of the radius of the inner boundary according to Eqs. 31 and 33 for the fully plastic slab. It will be observed that while the slab becomes fully plastic for $\lambda = 0.0034$, it can support a load $\lambda = 0.0085$ with sufficient plastic deformation. This is in marked contrast to the perfectly plastic slab which, in the present example, becomes fully plastic for $\lambda = 0.0033$ and can support no greater load.

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[6] The author wishes to thank Mr. R.K. Froyd for his assistance with the computations of the present section.

Figure 6 is derived from Figs. 4 and 5 and shows the circumferential stress and strain, respectively, at the inner boundary as functions of the load for the elastic, elastic-plastic, and fully plastic solutions. The discontinuity between the last two sections of the curve in each case is due to the change of assumptions between the two solutions. It is seen to be small in comparison with the other quantities involved.

References

1. P. Hodge and W. Prager, A variational principle for plastic materials with strain-hardening, J. Math. and Phys. 27, 1-10 (1948).

2. P. G. Hodge, Jr., On the plastic strains in slabs with cutouts, Paper No. 52 - A-22, to be published in J. Appl. Mech.

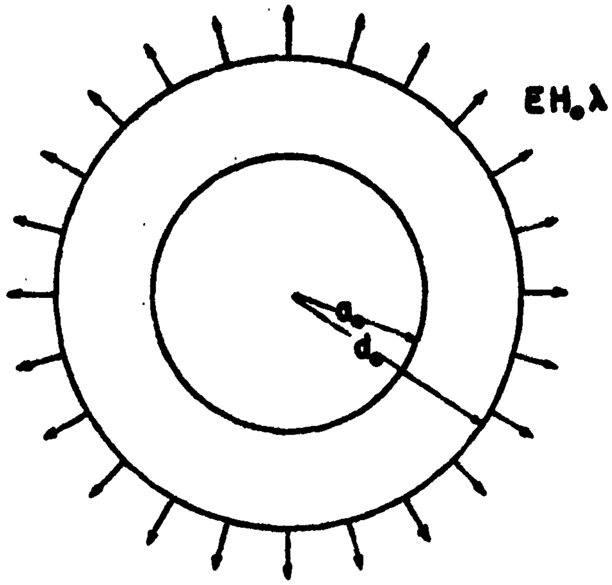

Fig. 1. Dimensions of Slab.

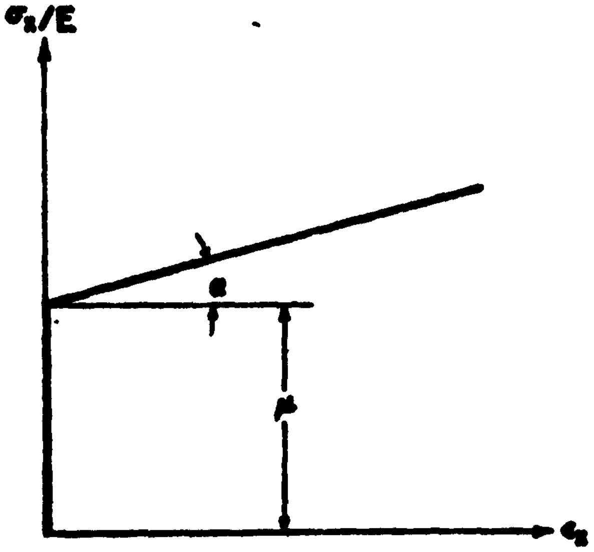

Fig. 3. Stress-Strain Curve for Large Strains.

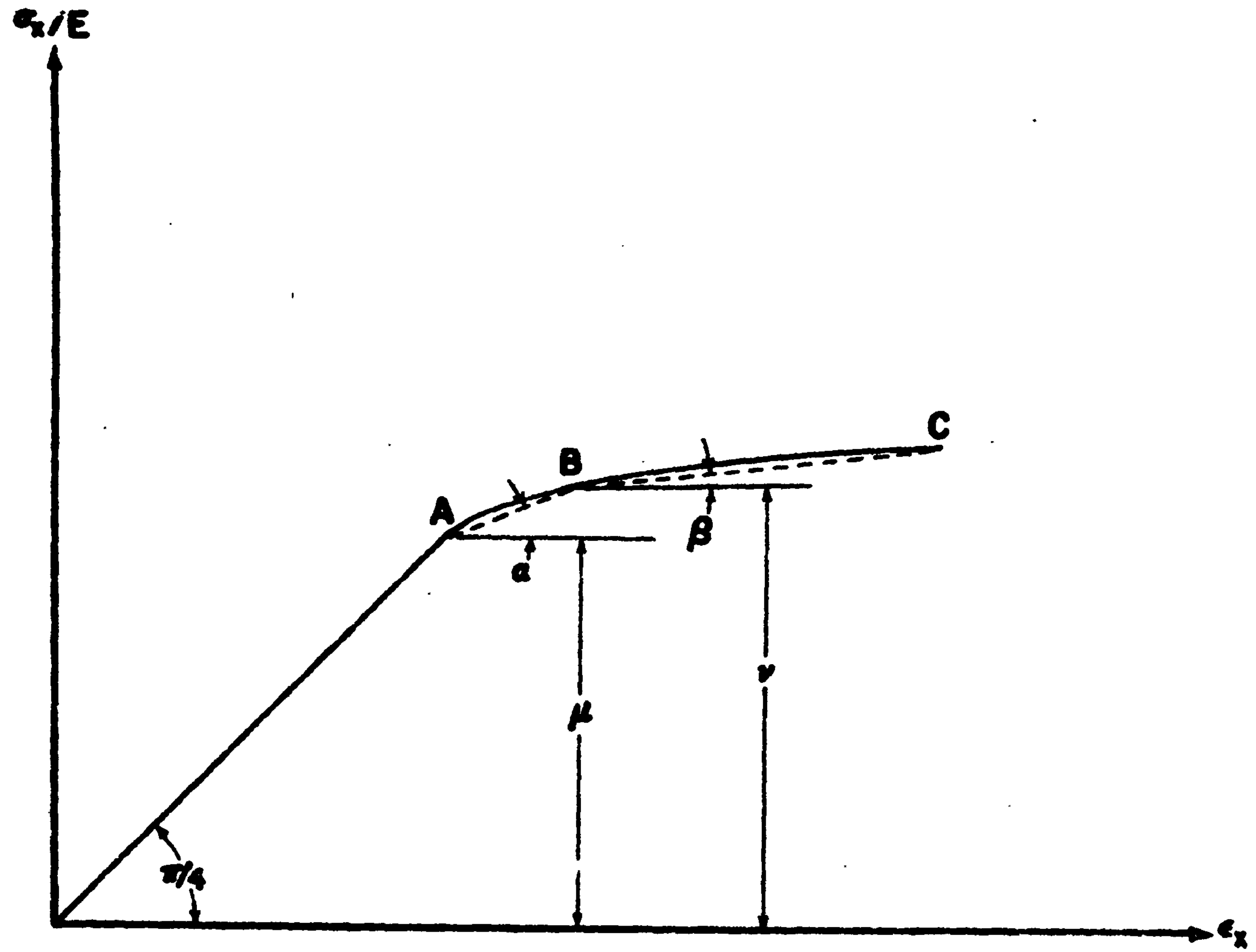

Fig. 2. Stress Strain Curve

——— Actual ------ Approximate

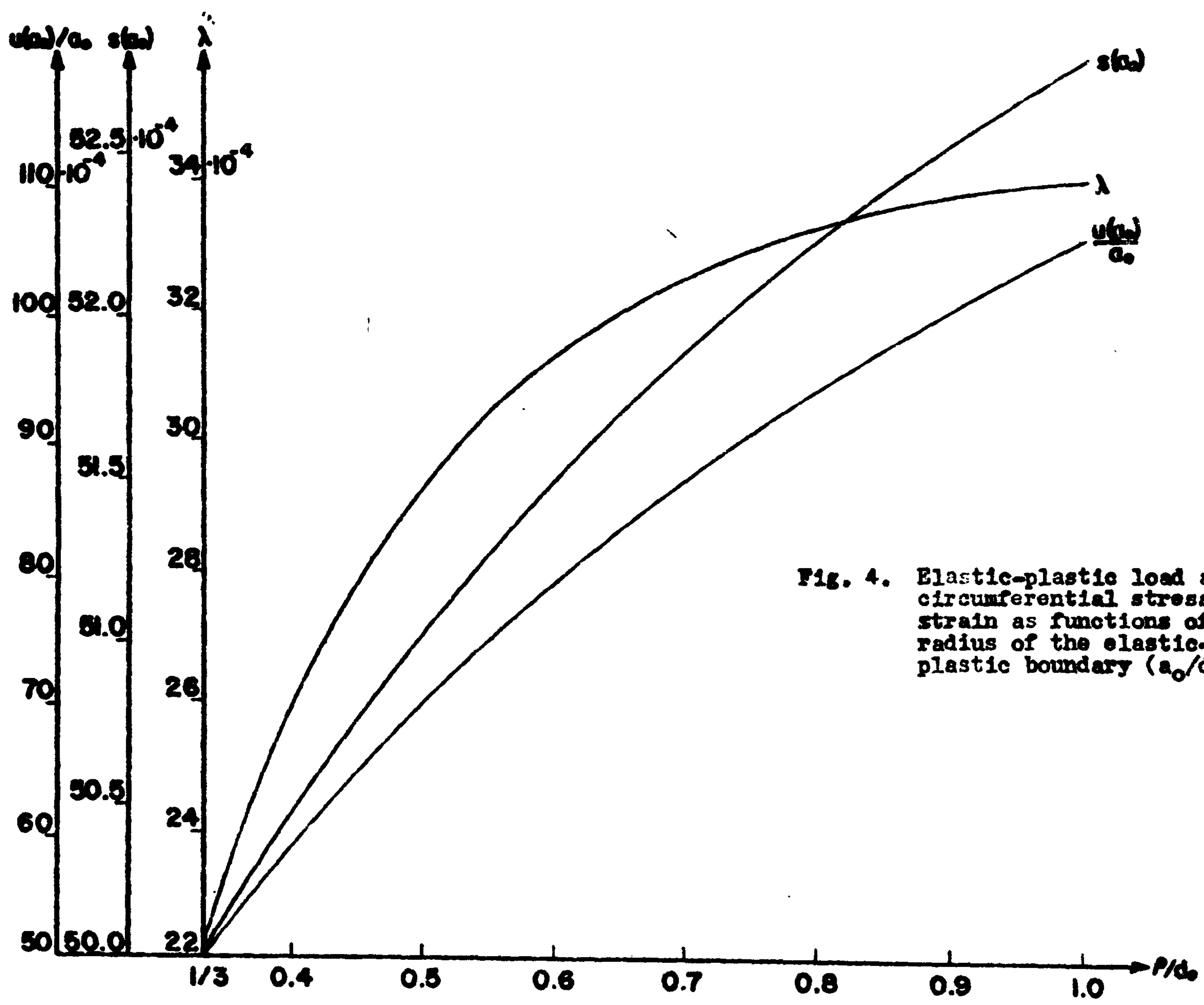

Fig. 4. Elastic-plastic load and circumferential stress and strain as functions of the radius of the elastic-plastic boundary ($a_o/d_o = 1/3$).

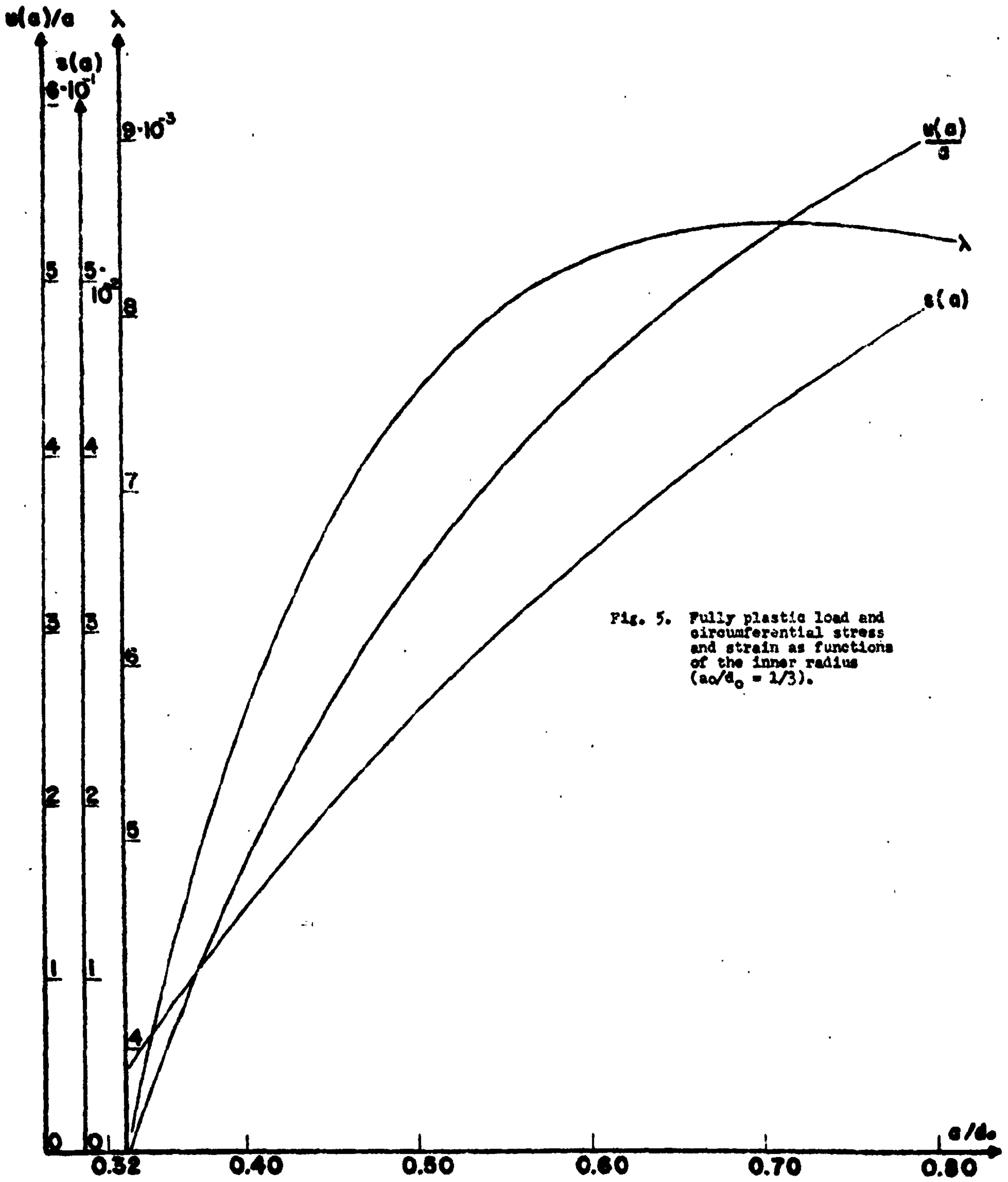

Fig. 5. Fully plastic load and circumferential stress and strain as functions of the inner radius ($a_0/d_0 = 1/3$).

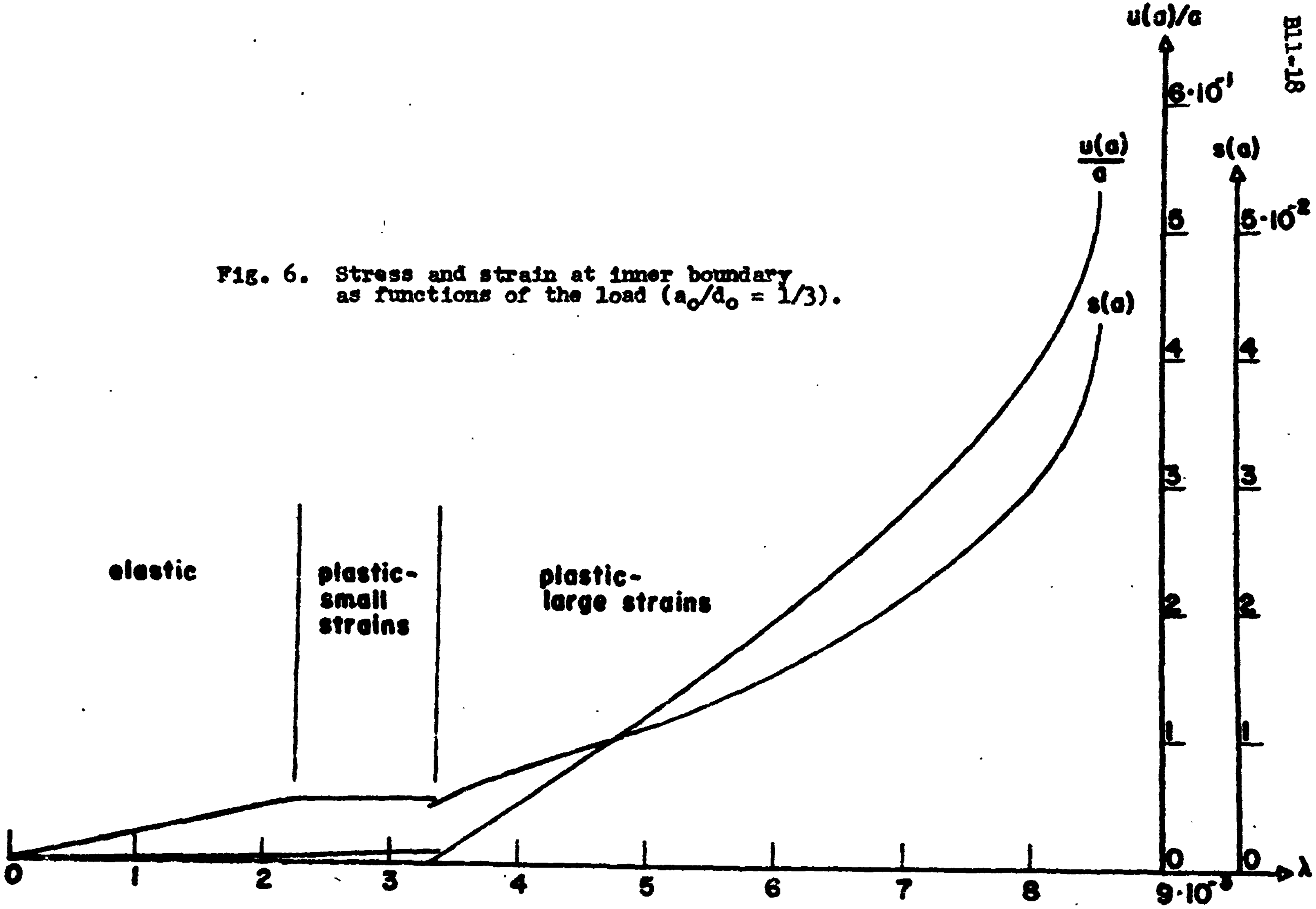

Fig. 6. Stress and strain at inner boundary as functions of the load ($a_o/d_o = 1/3$).